

without ambiguity to every possible development, however new, of mathematical language. It is susceptible only of a metaphysical definition as signifying the exercise, with regard to its content, of that faculty of the human mind whereby a multitude is capable of being regarded as an individual, or a complex as a monad. In a word, it is the symbol of individuality and unification." I am unable to assert that Sylvester foresaw the *phrasal forms* of modern APL 125 years ago, but his words seem remarkably apt in reference to these new developments.

Notation as a tool of thought

In ending I wish to quote from some of our great predecessors who appreciated the power of symbols as an aid to reasoning, or in Ken Iverson's memorable phrase, "notation as a tool of thought."

Lavoisier wrote a memoir in 1787 on the necessity of reforming the nomenclature of chemistry. In it he made this statement: "Languages are intended, not only to express by signs, as is commonly supposed, the ideas and images of the mind; but are also analytical methods, by the means of which, we advance from the known to the unknown, and to a certain degree in the manner of mathematicians. . . . Algebra is the analytical method by excellence [sic]; it has been invented to facilitate the operations of the understanding, and to render reasoning more concise, and to contract into a few lines what would have required whole pages of discussion; in fine, to lead, in a more agreeable and laconic method [*plus commode, plus prompt et plus sûre*], to the solution of the most complicated questions. Even a moment's reflection is sufficient to convince us that algebra is in fact a language: like all other languages it has its representative signs, its method and its grammar, if I may use the expression: thus an analytical method is a language; a language is an analytical method; and these two expressions are, in a certain respect synonymous [sic]."¹¹⁷

In 1821, Babbage, in his thought-provoking paper "On the Influence of Signs in Mathematical Reasoning," said: "The quantity of meaning compressed into small space by algebraic signs is a circumstance that facilitates the reasonings we are accustomed to carry on by their aid. The assumption of lines and figures to represent quantity and magnitude, was the method employed by the ancient geometers to present to the eye some picture by which the course of their reasonings might be traced: it was however necessary to fill up this out-

line by a tedious description, which in some instances even of no peculiar difficulty became nearly unintelligible, simply from its extreme length: the invention of algebra almost entirely removed this inconvenience, and presented to the eye a picture perfect in all its parts, disclosing at a glance, not merely the conclusion in which it terminated, but every stage of its progress. At first it appeared probable that this triumph of signs over words would have limits to its extent: a time it might be feared would arrive, when oppressed by the multitude of its productions, the language of signs would sink under the obscurity produced by its own multiplication. . . . Fortunately however such anticipations have proved unfounded.

"Examples of the power of a well-contrived notation to condense into small space a meaning which would—in ordinary language—require several lines, or even pages, can hardly have escaped the notice of most of my readers: in the calculus of functions, this condensation is carried to a far greater extent than in any other branch of analysis, and yet, instead of creating any obscurity, the expressions are far more readily understood than if they were written at length. . . . The power we possess by the aid of symbols of compressing into small compass the several steps of a chain of reasoning, whilst it contributes greatly to abridge the time which our enquiries would otherwise occupy, in difficult cases influences the accuracy of our conclusions: for from the distance which is sometimes interposed between the beginning and the end of a chain of reasoning, although the separate parts are sufficiently clear, the whole is often obscure. . . . The closer the succession between two ideas which the mind compares, provided those ideas are clearly perceived, the more accurate will be the judgement that results."¹¹⁸

"The advantage of selecting in our signs, those which have some resemblance to, or which from some circumstance are associated in the mind with the thing signified has scarcely been stated with sufficient force: the fatigue, from which such an arrangement saves the reader, is very advantageous to the more complete devotion of his attention to the subject examined; and the more complicated the subject, the more numerous the symbols and the less their arrangement is susceptible of symmetry, the more indispensable will such a system be found. This rule is by no means confined to the choice of the letters which represent quantity, but is meant to extend, when it is possible, to cases