Figure 10 Sylvester's 2 by 2 matrices

$$a + bi + cj + dk$$

$$i^2 = j^2 = k^2 = ijk = {}^{-}1$$

$$a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix} + c \begin{pmatrix} 0 & {}^{-}1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & -\theta \\ -\theta & 0 \end{pmatrix}$$

I   L   M   N

$$L^2 = M^2 = N^2 = LMN = -I$$

The matrices are:

```
    I=. 1 0,: 0 1

    s1=. 0 1,: 1 0                       [33.10]
    s2=. (0,-i),:(i,0)
    s3=. 1 0,: 0 _1
```

and the permutations are:

```
    z=. 0 1 2|.y=. s1;s2;s3

    I -:"2 p^"2 > y                       [33.9]
1 1 1

    f=. p-p^                              [33.11]
    g=. ({. f 1&|) -: (2*i)&* @ (2&|)
    I -:"0 g "3 > z
1 1 1

    f=. p : - @ p^                        [33.12a]
    g=. {. (> @ f) 1&|
    h=. g -:"2 i&* @ (2&|)
    1 1 -:"1 h "3 > z
1 1 1

    f=. p+p^                              [33.12b]
    g=. {. f (1&|)
    (0 0,:0 0) -:"2 g "3 > z
1 1 1
```

In each of these identities, function *f* describes the essential relationship; functions *g* and *h* make it possible to test all "cyclical permutations of the indices."[98]

After Sylvester returned to England, the principal exponents of the New Algebra in the United States were Benjamin Peirce and J. Willard Gibbs. Sylvester called Peirce's 1870 memoir[95] "a work which may almost be entitled to take rank as the 'Principia' of the philosophical study of the laws of algebraical operation." Gibbs's address "On Multiple Algebra" to the Section of Mathematics and Astronomy of the American Association for the Advancement of Science is a classic. In it Gibbs wrote the following:

"The multiple quantities corresponding to concrete quantities such as ten apples or three miles are evidently such combinations as ten apples + seven oranges, three miles northwestward + five miles eastward, or six miles in a direction 50 degrees east of north . . . . But if we ask what it is in multiple algebra which corresponds to an abstract number like twelve, which is essentially an operator, which changes one mile into twelve miles, and $1,000 into $12,000, the most general answer would evidently be: an operator which will work changes as, for example, that of ten apples + seven oranges into fifty apples and 100 oranges, or that of one vector into another. If the operation is distributive, it may not inappropriately be called multiplication, and the result is par excellence the product of the operator and the operand. The sum of operators, quâ operators, is an operator which gives for the product the sum of the products given by the operators to be added. The product of two operators is an operator which is equivalent to the successive operations of the factors."[101]