Figure 9 Sylvester's multiplication tables

COMPLEX NUMBERS

	1	i
1	1	i
i	i	−1

QUARTERIONS

	1	i	j	k
1	1	i	j	k
i	i	−1	k	−j
j	j	−k	−1	i
k	k	j	−i	−1

MATRIX MULTIPLICATION

	λ	μ	ν	π
λ	λ	μ	0	0
μ	0	0	λ	μ
ν	ν	π	0	0
π	0	0	ν	π

sion are to be selected—the multiplication table determines the basket into which their product is to be thrown. . . . The whole analytical side of the theory of quaternions merges into a particular case of the general theory of *Multiple Algebra*. As far as the present writer is aware, Professor Cayley in his Memoir on Matrices (1858), was the first to recognize the parallelism between quaternions and matrices . . . "[91]

Sylvester's locative symbols and multiplication tables for complex numbers, quaternions, and matrix multiplication are given in Figures 8 and 9 (from References 18, 91, 93, 94). By this method of representation Sylvester states in 1884: "a matrix is robbed as it were of its areal dimensions and represented as a linear sum." Sylvester's 2 by 2 matrices I, L, M, and N are given in Figure 10, where the matrices, "construed as complex quantities, are a linear transformation of the ordinary quaternion system 1, i, j, k." As he said: "Every matrix of the second order may be regarded as representing a quaternion, and vice versa."

Sylvester's matrix identities given in Figure 10 can be demonstrated very concisely in Iverson's J, which supports complex numbers. The inner product is given by *p*, and *square* computes the product of a matrix with itself; *i* is $\sqrt{-1}$. One line suffices to express the identities. The *match* function is -:

```
i=.%:_1
p=.+/ .*
square=.p~

I=. 1 0,: 0 1
L=. (i,0),:0, -i
M=. 0 _1,: 1 0
N=. (0,-i),: -i,0

(<-I)-:&.> (square &.> L;M;N),
        <L p M p N
```

```
┌─┬─┬─┬─┐
│1│1│1│1│
└─┴─┴─┴─┘
```

These matrices, derived by Sylvester (see also References 71, 95) as an exercise in pure mathematics, are intimately connected to the Pauli spin matrices, which have central significance in relativistic quantum theory; they are also close to the *spinor transformation*,[96] to *basis quaternions*, and the *basis elements* of the 16-dimensional Clifford numbers,[97] whose algebraic properties can easily be demonstrated in APL.[98–100] The three Pauli matrices (σ_1, σ_2, and σ_3) describing the spin of an electron, together with all permutations of Pauli's identities, can be stated formally and executed. These are shown below in J with the numbers in square brackets from Pauli.

Given:

```
p=. +/ .*
i=. %: _1
```